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SEMI-PARALLEL HYPERSURFACES OF A REAL SPACE FORM

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ABSTRACT

We show that a semi-parallel hypersurface of a sphere and a hyperbolic space is either flat, parallel or a rotation hypersurface whose profile curve is a helix.

0. Introduction

Semi-parallel submanifolds are defined as a generalization of parallel submanifolds, i.e. submanifolds with parallel second fundamental form. Parallel submanifolds M^n of a real space form $\widetilde{M}^{n+p}(c)$ have been classified by E. Backes and H. Reckziegel in [BR] and independently by M. Takeuchi in [T]. For arbitrary c and for $p=1$, the classification had already been done by H. B. Lawson, Jr. in [La]. For $c=0$ the classification was done by D. Ferus in [F1][F2] and in the case $p=1$ and $c=0$ by U. Simon and A. Weinstein in [SW].

Semi-parallel hypersurfaces of a Euclidean space have been classified by J. Deprez in [De1].

THEOREM 1: *[De1] Let M^n be a semi-parallel hypersurface of $\mathbb{E}^{n+1}$. Then there are three possibilities:*
(1) M^n is flat,
(2) M^n is parallel,
(3) M^n is a round cone, or a product of a round cone and a linear subspace.

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The same author has given a classification of semi-parallel surfaces of the Euclidean space [De2]. In [Lu] Ü. Lumiste has classified semi-parallel submanifolds of codimension 2 of the Euclidean space. In the present paper, we will give a classification of semi-parallel hypersurfaces of a real space form $\widetilde{M}^{n+1}(c)$, $c \neq 0$. Note that a semi-parallel hypersurface of a sphere is automatically a semi-parallel submanifold of codimension 2 of the Euclidean space. We give however an explicit description of those hypersurfaces without using the classification in [Lu]. Our aim is to prove the following theorem. In the formulation of the theorem we make use of the hypersurface model for a space form, i.e. we consider $\widetilde{M}^{n+1}(c)$, $c \neq 0$ as a hypersphere of $\mathbb{R}^{n+2}$, endowed with the Euclidean metric if $c > 0$ and with the Lorentzian metric if $c < 0$.

THEOREM 2: *Let M^n be a semi-parallel hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ with $c \neq 0$. Then there are three possibilities:*
(1) $n = 2$ and M^2 is flat,
(2) M^n is parallel,
(3) There exists a totally geodesic $\widetilde{M}^2(c)$, and a vector u in the linear subspace $\mathbb{R}^3$ of $\mathbb{R}^{n+2}$, containing $\widetilde{M}^2(c)$, such that M^n is a rotation hypersurface whose profile curve is a u-helix lying in $\widetilde{M}^2(c)$, and whose axis is $u^\perp$. Moreover, M^n is intrinsically isometric to a cone.

"Cone" is used in the sense of [Sz]. Rotation hypersurfaces and helices will be discussed in Section 2. This result has been announced in [Di], without details. A detailed description of the hypersurfaces (3) will be given here. We will also recall the classification of parallel hypersurfaces. Note that the round cones in Theorem 1 are rotation hypersurfaces of the type (3) of Theorem 2.

1. Preliminaries

Let M^n be an immersed hypersurface of a real space form $\widetilde{M}^{n+1}(c)$. We denote the metric on $\widetilde{M}^{n+1}(c)$ by $\langle\ ,\ \rangle$ and the Levi Civita connection of $(\widetilde{M}^{n+1}(c), \langle\ ,\ \rangle)$ by $\widetilde{\nabla}$. The induced metric on M^n is also denoted by $\langle\ ,\ \rangle$ and the Levi Civita connection of $(M^n, \langle\ ,\ \rangle)$ by ∇. Then we have the formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi,$$
$$\widetilde{\nabla}_X \xi = -SX,$$

where X and Y are tangent vector fields, ξ is a unit normal vector field, h and S denote respectively the second fundamental form, which is a symmetric $(0,2)$-

tensor field on M^n, and the shape operator of ξ, which is a $(1,1)$-tensor field on M^n. Note that h and S are related by $h(X,Y) = \langle SX, Y\rangle$.

M^n is called totally geodesic if $S = 0$. M^n is called totally umbilical if S is proportional to the identity transformation, say $S = \lambda I$. If $n > 1$, then λ is constant and M^n has constant curvature $c + \lambda^2$. A hypersurface is called parallel if $\nabla h = 0$ and is called semi-parallel if $R \cdot h = 0$, where $R \cdot h$ is the (0,4)-tensor field defined by

$$(R\cdot h)(X,Y,U,V) = (R(X,Y)\cdot h)(U,V) = -h(R(X,Y)U,V) - h(U,R(X,Y)V).$$

If M^n is parallel, then M^n is also semi-parallel.

A manifold M^n is called semi-symmetric if $R \cdot R = 0$, where $R \cdot R$ is the (1,5)-tensor field defined by

$$\begin{aligned}(R\cdot R)(X,Y,U,V,W) &= (R(X,Y)\cdot R)(U,V)W\\ &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W\\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W.\end{aligned}$$

Every locally symmetric manifold (i.e. satisfying $\nabla R = 0$) is semi-symmetric. Note that a parallel hypersurface is locally symmetric and a semi-parallel hypersurface is semi-symmetric. Semi-symmetric manifolds are classified by Z. Szabo in [Sz]. As examples of non-locally symmetric, semi-symmetric manifolds he obtains the so-called elliptic, Euclidean and hyperbolic cones. They are defined as follows. Let $M^{n-1}(c)$ be a real space form of constant curvature c. Let $I : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the identity function on $\mathbb{R}_0^+$. Then the warped product $M^n = \mathbb{R}_0^+ \times_I M^{n-1}(c)$ is called an elliptic, Euclidean or hyperbolic cone if $c > 0$, $c = 0$ or $c < 0$. For more information on warped products of manifolds, the reader is referred to [O'N]. We only recall the definition. Let M_1 and M_2 be Riemannian manifolds with metrics g_1 and g_2. Let M be the product manifold $M_1 \times M_2$ and let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the natural projections. If f is a positive real function on M_1, then we can define a Riemannian metric g on M^n by

$$g(X,Y) = g_1((\pi_1)_*X,(\pi_1)_*Y) + f(\pi_1(p))^2 g_2((\pi_2)_*X,(\pi_2)_*Y),$$

for $X, Y \in T_pM$. We call (M,g) the warped product of M_1 and M_2 with warping function f and denote this by $M_1 \times_f M_2$.

We now introduce the following model for the hyperbolic space. Let $H^{n+1}(c)$, $c < 0$, be the hypersurface of $\mathbb{R}^{n+2}$ given by

$$H^{n+1}(c) = \left\{x \in \mathbb{R}^{n+2} | x_1^2 + \cdots + x_n^2 + x_{n+1}^2 - x_{n+2}^2 = 1/c, x_{n+2} > 0\right\}.$$

If we endow $H^{n+1}(c)$ with the Riemannian metric induced by the Lorentzian metric

$$ds^2 = dx_1^2 + \cdots + dx_{n+1}^2 - dx_{n+2}^2$$

on $\mathbb{R}^{n+2}$, then $H^{n+1}(c)$ has constant negative curvature c. Then the totally umbilical hypersurfaces $H^{n+1}(c)$ are given by the intersection of $H^{n+1}(c)$ and an affine hyperplane α . The hypersurface is totally geodesic if α goes through the origin. If α does not pass through the origin, then the hypersurface is hyperbolic, parabolic or elliptic if the angle between α and e_{n+1} is smaller than, equal to or greater than $\pi/4$. We define a class of product hypersurfaces $S^k(c_1) \times H^{n-k}(c_2)$ with $c_1 > 0$, $c_2 < 0$ and $1/c_1 + 1/c_2 = 1/c$ by

$$S^k(c_1) \times H^{n-k}(c_2) = \left\{x \in H^{n+1}(c) | x_1^2 + \cdots + x_{k+1}^2 = 1/c_1\right\}.$$

These products, together with the totally umbilical hypersurfaces, are the only parallel hypersurfaces of $H^{n+1}(c)$, up to isometries of $H^{n+1}(c)$. We will denote in the following the metric ds^2 by $\langle \ , \ \rangle$.

Now let $S^{n+1}(c)$, $c > 0$, be the hypersphere of radius $1/c$ of $\mathbb{R}^{n+2}$, centered at the origin, i.e.

$$S^{n+1}(c) = \left\{x \in \mathbb{R}^{n+2} | x_1^2 + \cdots + x_{n+1}^2 + x_{n+2}^2 = 1/c\right\}.$$

Then any totally umbilical hypersurface of $S^{n+1}(c)$ is given by the intersection of $S^{n+1}(c)$ and an affine hyperplane α and it is totally geodesic if α goes through the origin. Again we define a class of product hypersurfaces $S^k(c_1) \times S^{n-k}(c_2)$ with $c_1 > 0$, $c_2 > 0$ and $1/c_1 + 1/c_2 = 1/c$ by

$$S^k(c_1) \times S^{n-k}(c_2) = \left\{x \in S^{n+1} | x_1^2 + \cdots + x_{k+1}^2 = 1/c_1\right\}.$$

These products are together with the totally umbilical hypersurfaces the only parallel hypersurfaces of $S^{n+1}(c)$, up to isometries of $S^{n+1}(c)$.

The totally umbilical hypersurfaces of the Euclidean space $\mathbb{E}^{n+1}$ are the affine hyperplanes, which are totally geodesic, and the ordinary hyperspheres. The parallel hypersurfaces of $\mathbb{E}^{n+1}$ are the hyperplanes and the hyperspheres and products of affine subspaces and spheres.

The same statements can be made for totally geodesic and totally umbilical submanifolds of greater codimension. In general, the intersection of $S^{n+1}(c)$ and $H^{n+1}(c)$ with affine subspaces are called spheres, and they are the only totally umbilical submanifolds. Proofs of the given classification can be found in [C] and [La].

Since S is symmetric, there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM for every $p \in M^n$ consisting of eigenvectors of S, i.e. $Se_i = \lambda_i e_i$. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the principal curvatures of M^n at p. If a basis $\{e_1, e_2, \ldots, e_n\}$ occurs in the following, it will always mean a basis of eigenvectors of S. Then the equation of Gauss states that

$$R(e_i, e_j) = \nu_{ij}\, e_i \wedge e_j\,, \tag{1.1}$$

where $\nu_{ij} = c + \lambda_i\lambda_j$ and $\wedge$ associates to two vectors $X, Y \in T_pM$ an endomorphism $X \wedge Y$ of T_pM by

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

According to [R, lemma 2.1] there exist continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_n$ on M^n, such that for every $p \in M^n$, $\lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p)$ are the eigenvalues of S_p. The following proposition can be proved straightforwardly.

PROPOSITION 2.1: *For a hypersurface M^n of $\widetilde{M}^{n+1}(c)$ the following conditions are equivalent:*

(1) *$R \cdot h = 0$, i.e. M^n is semi-parallel,*

(2) *at each point $p \in M^n$ the shape operator has the following form:*

$$S_p = \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{pmatrix},$$

with $\lambda\mu + c = 0$, or $\lambda = \mu$.

2. Rotation Hypersurfaces and Helices

Rotation hypersurfaces: In this section we briefly recall what is a rotation hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ with $c \neq 0$ following [CD]. We always consider $\widetilde{M}^{n+1}(c)$ as a hypersphere in $(\mathbb{R}^{n+2}, ds^2)$. Let P^3 be a 3-dimensional linear subspace linear space of $\mathbb{R}^{n+2}$ that intersects $\widetilde{M}^{n+1}(c)$. We denote the intersection by $\widetilde{M}^2(c)$; if $c < 0$ we take only the upper part. Let P^2 be any linear subspace in P^3. We recall that any isometry of $\widetilde{M}^{n+1}(c)$ is the restriction to $\widetilde{M}^{n+1}(c)$ of an orthogonal transformation of $(\mathbb{R}^{n+2}, ds^2)$, and conversely. Let

$O(P^2)$ be the group of orthogonal transformations (with positive determinant) that leave P^2 pointwise fixed. We take any curve α in $\widetilde{M}^2(c)$ which does not intersect P^2. The orbit of α under $O(P^2)$ is called the rotation hypersurface with profile curve α and axis P^2. The orbit of $\alpha(s)$ for a fixed s is a sphere, and if $c < 0$, then this sphere is elliptic, hyperbolic or parabolic according to P^2 respectively being Lorentzian, Riemannian or degenerate.

In order to give a parametrization of a rotation hypersurface of the different types, we introduce the vector $u \in P^3$ such that P^2 coincides with $u^\perp = \{v \in P^3 \mid \langle v, u\rangle = 0\}$. We can always assume that u has length 1, -1 or 0, according to P^2 respectively being Lorentzian, Riemannian or degenerate, and that $\langle u, \alpha\rangle > 0$. Let $\delta = \langle u, u\rangle$. We define the map Q as the orthogonal projection of P^3 on $u^\perp$ if $\delta \neq 0$ and as the identity map of P^3 if $\delta = 0$. Further, let P^{n-1} be the orthogonal complement of P^3 in $\mathbb{R}^{n+2}$ and let P^n be the linear space, spanned by P^{n-1} and u. If $\delta = 1$, resp. $\delta = -1$, then P^n is Riemannian, resp. Lorentzian, and we can define a mapping ϕ of $M^{n-1}(\delta)$ into P^n by considering $M^{n-1}(\delta)$ as a unit hypersphere in P^n. If $\delta = 0$, then we can define a mapping ϕ of $M^{n-1}(0)$ into P^n by identifying $M^{n-1}(0)$ and P^{n-1} and defining

$$\phi(p) = p - \frac{1}{2}\langle p, p\rangle u.$$

Then a parametrization of the rotation hypersurface of α around the axis P^2 is given by

$$f(s, p) = Q(\alpha(s)) + \langle \alpha(s), u\rangle \phi(p).$$

If we assume that s is the arc length of α, then it follows immediately that the rotation hypersurface M^n is intrinsically the warped product $U \times_\rho M^{n-1}(\delta)$ where U is an open interval of $\mathbb{R}$ and ρ is defined by $\rho(s) = \langle \alpha(s), u\rangle$, see [CD, (3.9)].

The second fundamental form of M^n is given by

$$h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = \frac{\rho'' + c\rho}{\sqrt{\delta - c\rho^2 - \rho'^2}} \tag{2.1}$$

and

$$h(X, Y) = -\frac{\sqrt{\delta - c\rho^2 - \rho'^2}}{\rho}\langle X, Y\rangle \tag{2.2}$$

for X and Y tangent to $M^{n-1}(\delta)$, see [CD,(3.10)].

Helices: Let $\mathbb{R}^3$ be endowed with the Euclidean or Lorentzian metric and let u be any vector in $\mathbb{R}^3$. Let α be a nondegenerate curve in $\mathbb{R}^3$, i.e. $\langle \alpha', \alpha' \rangle \neq 0$. We then can assume that α is parametrized by arc length. We say that α is a u-helix if $\langle \alpha', u \rangle$ is constant.

Let $\widetilde{M}^2(c)$ be a hypersphere in $\mathbb{R}^3$ like above and let α be a u-helix which lies on $\widetilde{M}^2(c)$. Then we can give a parametrization of α as follows.

Case 1: $c > 0$. In this case we have a helix in Euclidean space, lying on a sphere. These curves are also considered in [BL, p. 49]. We can assume that $u = e_1$ and that α is not a circle in a plane $x_1 = D$. Then α has as arc length parametrization :

$$\begin{aligned} \alpha_1(s) &= as, \\ \alpha_2(s) &= \left(1/c - a^2 s^2\right)^{\frac{1}{2}} \sin(\psi(s)), \\ \alpha_3(s) &= \left(1/c - a^2 s^2\right)^{\frac{1}{2}} \cos(\psi(s)), \end{aligned}$$

where $0 < a < 1$ and ψ is defined by

$$\psi(s) = \frac{1}{a} \int_0^{as} \frac{\left((1-a^2)/c - t^2\right)^{\frac{1}{2}}}{1/c - t^2}\, dt.$$

Note that α is defined on $U = \left]0, \left(\frac{1-a^2}{ca^2}\right)^{\frac{1}{2}}\right[$.

Case 2: $c < 0$ and $\langle u, u \rangle > 0$. Here we can assume that $u = e_1$. We suppose that α is not a Lorentzian circle in a plane $x_1 = D$. Then α has an arc length parametrization of the form:

$$\begin{aligned} \alpha_1(s) &= as, \\ \alpha_2(s) &= \left(a^2 s^2 - 1/c\right)^{\frac{1}{2}} \sinh(\psi(s)), \\ \alpha_3(s) &= \left(a^2 s^2 - 1/c\right)^{\frac{1}{2}} \cosh(\psi(s)), \end{aligned}$$

where $a > 0$ and ψ is defined by

$$\psi(s) = \frac{1}{a} \int_{aB}^{as} \frac{\left(t^2 + (a^2-1)/c\right)^{\frac{1}{2}}}{t^2 - 1/c}\, dt.$$

Here $B = 0$ if $a \leq 1$ and $B = \left((1-a^2)/ca^2\right)^{1/2}$ if $a > 1$ and α is defined on $U = \left]B, \infty\right[$.

Case 3: $c < 0$ and $\langle u, u\rangle = 0$. In this case we choose coordinates on $\mathbb{R}^3$ such that $u = e_2$ and such that the metric on $\mathbb{R}^3$ takes the form $2dx_1dx_2 + dx_3^2$. Then α takes the form

$$\begin{aligned}
\alpha_1(s) &= s,\\
\alpha_3(s) &= -\left(s^2 + 1/c\right)^{\frac{1}{2}} + s\ln\left((-c)^{\frac{1}{2}}s + (-cs^2-1)^{\frac{1}{2}}\right),\\
\alpha_2(s) &= \left(1/c - \alpha_3^2\right)/(2s).
\end{aligned}$$

Here α is defined on $U = \,](1/(-c))^{1/2}, \infty[$.

Case 4: $c < 0$ and $\langle u, u\rangle < 0$. In this case we can assume that $u = e_3$. Then either α is a circle in a plane $x_3 = D$, or

$$\begin{aligned}
\alpha_3(s) &= as,\\
\alpha_1(s) &= \left(a^2s^2 + 1/c\right)^{\frac{1}{2}}\sin(\psi(s)),\\
\alpha_2(s) &= \left(a^2s^2 + 1/c\right)^{\frac{1}{2}}\cos(\psi(s)),
\end{aligned}$$

where $a > 0$ and ψ now is defined by

$$\psi(s) = \frac{1}{a}\int_{aB}^{as}\frac{\left(t^2 + (a^2+1)/c\right)^{\frac{1}{2}}}{t^2 + 1/c}\,dt.$$

Here $B = \left((1+a^2)/(-ca^2)\right)^{1/2}$ and α is defined on $U = \,]B, \infty[$.

Rotating helices: . We consider a linear subspace P^3 in $\mathbb{R}^{n+2}$ like above, take a u-helix α on $\widetilde{M}^2(c)$ and rotate it around $P^2 = u^{\perp}$. Therefore we have to assume that $\langle \alpha, u\rangle \neq 0$. Then the corresponding function $\rho = \langle \alpha, u\rangle$ satisfies $\rho = rs + t$, where r and t are constant real numbers that don't vanish simultaneously. Then it follows from (2.1) and (2.2) that the shape operator takes the form (2) of Proposition 2.1. Hence we have a semi-parallel rotation hypersurface. Moreover, we can notice that either M^n is a cone, if $r \neq 0$, or M^n is the product of an open interval U and a space of constant curvature $M^{n-1}(\delta)$, if $r = 0$. In this last case, both principal curvatures are constant, so M^n has parallel second fundamental form.

3. Proof of the Theorem

So let M^n be a hypersurface of $\widetilde{M}^{n+1}(c)$ that satisfies $R \cdot h = 0$. Since the principal curvature functions are continuous, it follows that U, defined by $U =$

$M_1 \cup M_2$, whereby $M_1 = \{p \in M^n \mid \lambda \neq \mu\}$ and $M_2 = \operatorname{int}\{p \in M^n \mid \lambda = \mu\}$, is an open dense subset of M^n. Then $\lambda = \mu$ is a constant function on M_2 and both λ and μ are differentiable on M_1. If M_1 is empty, then M^n is totally umbilical and consequently M^n is parallel. If $n = 2$, then $\lambda\mu + c = 0$ implies that M_1 is flat. Since the Gaussian curvature is continuous and constant on both M_1 and M_2, we obtain that M^2 is flat. Hence we assume that M_1 is not empty, and we restrict our attention to an arbitrary connected component of M_1, also denoted by M_1. We recall that on M_1 it holds that $\lambda\mu + c = 0$.

If $m_\lambda > 1$ and $m_\mu > 1$, then it is easy to show that both λ and μ are constant and that M_1 is parallel; cf. [R, Proposition 4.4]. So we suppose that $m_\lambda = 1$ and $m_\mu = n - 1 > 1$.

Then [CD, theorem 4.2] implies that, at least locally, M_1^n is a rotation hypersurface. The profile curve α (an integral curve of T_λ) is a curve lying in a 2-dimensional totally geodesic subspace $\widetilde{M}^2(c)$ of $\widetilde{M}^{n+1}(c)$. Let P^3, P^2, u and ρ be like in section 2. Now (2.1) and (2.2) imply that $\rho'' = 0$. Hence $\langle u, \alpha(s)\rangle = As + B$, where $A \geq 0$ and B are constant real numbers and s is the arc length of α . Thus α is a u-helix in P^3.

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